Scientific report on EPSRC Grant EP/D06,1997/1 Self-correspondences of K3 surfaces via moduli of sheaves

by Principal Investigator
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1 Introduction

This grant is devoted to self-correspondences of K3 surfaces via moduli of sheaves.

It continued 3 years. During 1st year, October 2006 – October 2007, it was performed by me together with Dr. Antonio Rapagnetta. During remaining 2 years, May 2008 – February 2010, it was performed by me together with Dr. Timothy Logvinenko. For 1 month in July – August of 2009 I invited my old collaborator on the subject Dr. Carlo Madonna.

2 Results

In series of our papers with Carlo Madonna [11], [12] and [24], [25], we considered self-correspondences of algebraic K3 surfaces X (over $\mathbb{C}$) via moduli of sheaves.

They are determined by primitive isotropic Mukai vectors $v = (r, H, s)$ with $r \in \mathbb{N}$, $s \in \mathbb{Z}$ and $H \in N(X)$ where $N(X)$ is the Picard lattice of X, and $H^2 = 2rs$. Due to Mukai [14]—[15] (see also Yoshioka [43]), the moduli space $Y = M_X(v)$ of stable (with respect to some ample element $H' \in N(X)$) coherent sheaves on X of rank r with first Chern class H and Euler characteristic $r + s$ is again a K3-surface. Chern class of the corresponding quasi-universal sheaf $\mathcal{E}$ on $X \times Y$ gives some 2-dimensional algebraic cycle on $X \times Y$ and can be considered as a correspondence between X and Y. If $Y \cong X$, it can be considered as a self-correspondence of X via moduli of sheaves with Mukai vector v.

In our papers above, we studied when $Y \cong X$, and we gave a complete answer if the Picard number of X is 1 or 2, and X is general for its Picard lattice.

2.1 The general results of [25]

During grant period, in 2007, my final paper [25] was published. It contained the general answer to this problem, for arbitrary primitive isotropic Mukai vectors and arbitrary Picard lattices $N(X)$ of the rank $\rho(X) = \mathrm{rk}\, N(X) = 1$ or 2. It contained very difficult calculations, its volume was 51 page. In [11], we assumed that $v = (2, H, 2)$ where $H^2 = 8$. In [12], we assumed that $v = (a, H, a)$ where $H^2 = 2a^2$. In [19], we assumed that $H \cdot N(X) = \mathbb{Z}$. It is surprising to me that it was possible to answer this question in general which was done in [25]. But, the particular cases considered in [11], [12] and [24] are very classical and important. Moreover, some results of these papers were more exact and simpler than in general.

We mention that the case of $\rho(X) = 2$ is especially important because it *describes all divisorial conditions on moduli of K3 surfaces X when the moduli of sheaves Y are isomorphic to X.*

2.2 The explicit results of [13]

Unfortunately, the results of [11], [12] and [24], [25] were not explicit. To prove existence of the isomorphism $Y \cong X$, we used Global Torelli Theorem for K3 surfaces by Piatetsky-Shapiro and Shafarevich [33]. We compared periods of K3 surfaces X and Y which was enough to prove that $X \cong Y$, by the Global Torelli Theorem. Unfortunately, this does not give any explicit geometric/algebraic construction of the isomorphism.

In our paper with Carlo Madonna [13] (published in 2008), we showed that results of [11], [12] and [24], [25] can be made explicit. We showed that when [11], [12] and [24], [25] gave existence of an isomorphism between X and $Y = M_X(v)$, then there exists their isomorphism which is a composition of some standards isomorphisms: Tensor products T_D by linear bundle D, Mukai reflections δ, the isomorphisms $\nu(d_1, d_2)$ between moduli of sheaves over X, and Tyurin's isomorphism Tyu between moduli of sheaves over X and X itself. All these isomorphisms exist for general (with Picard number one) K3 surfaces, and can be considered as "standard".

2.3 Negative and positive results of [26]

In [26] we tried to find an approach to the same problem for arbitrary Picard number $\rho = \rho(X)$.

We showed that there exists a 3-dimensional hyperbolic lattice N, $H \in N$ and a type (r, H, s) of primitive isotropic Mukai vector (thus, $H^2 = 2rs$) such that for any K3 surface X with $N \subset N(X)$ (thus, $\mathrm{rk}\, N(X) \geq \mathrm{rk}\, N = 3$) the muduli $Y = M_X(v)$ are isomorphic to X, but the same is not valid for any K3 surface X such that $H \in N(X) \subset N$ and $\rho(X) = \mathrm{rk}\, N(X) \leq 2$. Thus, the isomorphism $Y = M_X(v) \cong X$ is not a specialisation of an isomorphism which comes from divisorial condition on moduli of K3 surfaces. Actually, we showed that there are plenty of similar examples. The cunstruction heavily uses our results for Picard number 2 above, and it is very non-trivial. This construction also shows that there are plenty of similar examples.

These examples show that results for arbitrary Picard number cannot be obtained by the direct specialisation to our results for Picard number 2 above.

In [26] we also introduced *action* on $H^2(X,\mathbb{Q})$ of a self-correspondence of a K3 surface X on the moduli of sheaves with primitive isotropic Mukai vector $v = (r, H, s)$ of X. Moreover, we observed that a self-correspondence of Tyurin's type [36]—[38] (it is defined by a primitive isotropic Mukai vectors $(\pm H^2/2, H, \pm 1)$ where $\pm H^2 > 0$) acts as reflection in $H^2(X,\mathbb{Q})$ with respect to H.

2.4 Results of [27] for arbitrary Picard number

Using results of [26], for K3 surfaces with arbitrary Picard number, we showed that self-correspondences of K3 surfaces via moduli of sheaves with primitive isotropic Mukai vector are numerically equivalent to compositions of self-correspondences of Tyurin's type, (-2)-roots, and some finite number of self-correspondence via moduli of sheaves with primitive isotropic Mukai vector. In particular, $Y = M_X(v)$ is isomorphic to X if and only if its action on $H^2(X,\mathbb{Z})$ is numerically equivlent to such a composition.

One can consider this result as a functorial and general (for arbitrary Picard number) description of self-correspondences of K3 surfaces via moduli of sheaves with primitive isotropic Mukai vectors.

2.5 Results of [27] and arithmetic hyperbolic reflection groups

A hyperbolic lattice (i. e. an integral symmetric bilinear form of sinature $(1,n)$) is called *reflective* if its automorphism group is generated by reflections with respect to negative roots of the lattices. Equivalently, the reflection group of the integral lattice is *arithmetic hyperbolic reflection group.*

As application, in [27] we characterized K3 surfaces with reflective Picard lattices: A K3 surface X has reflective Picard lattice if and only if actions of negative Tyurin's self-correspondences with integral action in Picard lattice and (-2)-roots generate a subgroup of finite index in the automorphism group (integral) of the Picard lattice.

This can be considered as similar to the classical result by Piatetsky-Shapiro and Shafarevich [33]: A K3 surface has finite automorphism group if and only if its Picard lattice is 2-reflective (i. e., its automorphism group is generated, up to finite index, by reflections with respect to (-2)-roots).

In our old papers [19]—[22] we showed that the number of reflective hyperbolic lattices is finite, in essential, for each fixed rank. Vinberg in [40]–[42] showed that their rank is also bounded from above by 30. For K3 surfaces, the rank of Picard lattice cannot be more than 22 in any characteristic of the basic field.

As application, we obtain that except finite number of cases, in essential, Tyurin's self-correspondences with integral action and (-2) roots generate a subgroup of infinite index in the automorphism group of the Picard lattice of a K3 surface, and all such exceptions can be found. For example, in our old papers [19], [22] all 2-reflective integral hyperbolic lattices were found.

2.6 Results of [28]—[32] about classification of arbitrary reflective hyperbolic lattices and arithmetic hyperbolic reflection groups

Results above showed that self-correspondences of K3 surfaces via moduli of sheaves are closely related to arithmetic hyperbolic reflection groups. The general class of reflective lattices and arithmetic hyperbolic reflection groups is actually much wider. By Vinberg, [39], arithmetic hyperbolic reflection groups are subgroups of finite index of reflective hyperbolic lattices. In general, they are reflective hyperbolic lattices over rings of integers of totally real algebraic number fields (the ground fields of these groups). There description and classification includes, in particular, classification of maximal arithmetic hyperbolic reflection groups, and it is important to the theory of hyperbolic Lie algebras. Without any doubts, they are also important for self-correspondences of appropriate K3 surfaces which will be an important subject for the future. In our old papers [19]–[22] (the last paper was my report on the International Congress of Mathematicians at Berkeley in 1986 which summarized these results) it was shown that for the fixed main invariants n which is the dimension of the hyperbolic space, and N which is the degree of the ground field over $\mathbb{Q}$, the number of reflective hyperbolic lattices and the corresponding maximal arithmetic reflection groups is finite. Moreover, N has some absolute upper bound $N(14)$ if the dimension $n \geq 10$. In 1981, Vinberg showed that $n < 30$.

Existence of an upper bound for N when $n \leq 9$ was not known for a long time. Only in 2005, it was shown in [7] for $n = 2$, and in [1] for $n = 3$. In [28], I showed that results and methods of my old papers are strong enough to close the remaining gap for $4 \leq n \leq 9$ by a very short consideration. Thus, finally, in 2006, during the time of my grant, finiteness became known for the general class of reflective hyperbolic lattices over all possible fields, and for all arithmetic hyperbolic reflection groups.

To get an effective finiteness, one had to find an effective upper bound for the degree N which was not known. Using our methods, we showed in [29] that $N \leq 56$ for $n \geq 6$; in [30], that $N \leq 138$ for $n = 4$, 5; in [31], that $N \leq 909$ for $n = 3$ (it also gave another proof of finiteness for $n = 3$); in [29] that $N \leq 44$ for $n = 2$. Recently, these results were significantly improved in our last preprint [32]: it was shown that $N \leq 25$ for $n \geq 6$; $N \leq 44$ for $n = 3, 4$, and 5 (it can be improved to $N \leq 35$ using recent result of [2] and [8]). In [8], Prof. Maclachlan showed that $N \leq 11$ for $n = 2$.

These results will be important for further enumeration of the finite number of reflective hyperbolic lattices, and maximal arithmetic hyperbolic reflection groups over all fields together.

Without any doubts, these results will find applications to appropriate self-correspondences of K3 surfaces, and they will find important applications in Mathematics. These are important subjects for future investigations.

2.7 Results of [35] by A. Rapagnetta

One should expect that our results above about self-correspondences of K3 surfaces can be generalized to irreducible symplectic varieties. The most known are Hilbert schemes of points on a K3 surface, and moduli of sheaves over a K3 surface with not necessarily isotropic Mukai vector. Some different types of irresucible symplectic varieties were found by O'Grady. To transfer our results to these varieties, the most important invariant of these varieties is so called Beauville (or Beauville–Bogomolov) form which is similar to the 2-dimensional cohomology lattice of K3 surfaces. In [35] (see also [34]), Antonio Rapagnetta calculated Beauville form and other important invariants for 10-dimensional O'Grady irreducible symplectic variety. This gives a hope that our results about self-correspondences of K3 surfaces can be transformed to these varieties.

2.8 Results of [9] and [10] by Carlo Madonna

Papers by Carlo Madonna [9] and [10] give some examples of possible generalizations of our results about self-correspondences of K3 surfaces via moduli of sheaves to some irreducible symplectic varieties related to K3 surfaces.

Without any doubts, this is one of the most interesting future subjects related to the results and methods developed in this grant.

2.9 Results of [3]— [6] by Timothy Logvinenko

During the grant time, Dr. Timothy Logvinenko pusblished and submitted several important papers and preprints mainly related to derived categories of sheaves and McKay correspondence for finite subgroups of $GL_n(C)$. See [3]—[6]. Since these results use derived categories of sheaves, they are closely related to moduli of sheaves over K3 surfaces and self-correspondences of K3 surfaces via moduli of sheaves.

As we have seen above, Tyurin's self-correspondences of K3 surfaces are very important for description of all self-correspondences of K3 surfaces via moduli of sheaves. A. Tyurin gave a geometric description of these self-correspondences in "general" cases. Results and methods of [3]—[6] give a hope that a similar explicit geometric description can be obtained for Tyurin's self-correspondences of K3 surfaces in all cases. It will be very important for further geometric applications of our results.

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